4.4: Simplification of Grammars

In this section, we say what it means for a grammar to be simplified, give a simplification algorithm for grammars, and see how to use this algorithm in Forlan.

Motivating Example

Suppose G is the grammar

$$\begin{aligned} \mathsf{A} &\to \mathsf{BB1}, \\ \mathsf{B} &\to \mathsf{0} \mid \mathsf{A} \mid \mathsf{CD}, \\ \mathsf{C} &\to \mathsf{12}, \\ \mathsf{D} &\to \mathsf{1D2}. \end{aligned}$$

Question: what is odd about this grammar?

Answer: First, D doesn't generate anything.

Second, there is no valid parse tree that starts at G's start variable, A, has a yield that is in $\{0, 1, 2\}^* =$ **alphabet** G, and makes use of C.

Reachable, Generating and Useful Variables

Suppose G is a grammar. We say that a variable q of G is:

- *reachable in G* iff there is a $w \in \textbf{Str}$ such that w is parsable from s_G using G, and $q \in \textbf{alphabet}\, w$;
- *generating in G* iff there is a $w \in \textbf{Str}$ such that q generates w using G, i.e., w is parsable from q using G, and $w \in (\textbf{alphabet}\, G)^*$;
- *useful in G* iff q is both reachable and generating in G.

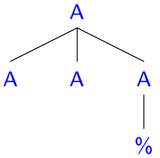
Redundant Productions

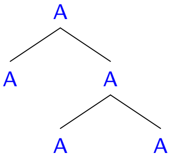
Now, suppose H is the grammar

$$\mathsf{A} \to \% \mid 0 \mid \mathsf{AA} \mid \mathsf{AAA}.$$

What is odd about this grammar?

Here, the productions $\mathsf{A} \to \mathsf{AA}$ and $\mathsf{A} \to \mathsf{AAA}$ are redundant, although only one of them can be removed:





Redundant Productions

Given a grammar G and a finite subset U of $\{\,(q,x) \mid q \in Q_G$ and $x \in \mathbf{Str}\,\}$, we write G/U for the grammar that is identical to G except that its set of productions is U.

If G is a grammar and $(q,x) \in P_G$, we say that:

- (q,x) *is redundant in* G iff x is parsable from q using H, where $H = G/(P_G - \{(q,x)\})$; and
- (q,x) *is irredundant in* G iff (q,x) is not redundant in G.

Simplified Grammars

A grammar G is *simplified* iff either

- every variable of G is useful, and every production of G is irredundant; or
- $|Q_G| = 1$ and $P_G = \emptyset$.

Proposition 4.4.1

If G is a simplified grammar, then $\textbf{alphabet}\ G = \textbf{alphabet}(L(G))$.

Simplified Grammars

Proof. Suppose $a \in \textbf{alphabet}\, G$. We must show that $a \in \textbf{alphabet}\, w$ for some $w \in L(G)$.

We have that every variable of G is useful, and there are $q \in Q_G$ and $x \in \textbf{Str}$ such that $(q, x) \in P_G$ and $a \in \textbf{alphabet}\, x$.

Thus x is parsable from q. Since every variable occurring in x is generating, we have that q generates a string x' containing a.

Since q is reachable, there is a string y such that y is parsable from s_G, and $q \in \textbf{alphabet}\, y$. Since every variable occurring in y is generating, there is a string y' such that y' is parsable from s_G, and q is the only variable of $\textbf{alphabet}\, y'$.

Putting these facts together, we have that s_G generates a string w such that $a \in \textbf{alphabet}\, w$, i.e., $a \in \textbf{alphabet}\, w$ for some $w \in L(G)$. $\square$

Algorithm for Removing Redundant Productions

Given a grammar G, $q \in Q_G$ and $x \in \mathbf{Str}$, we say that (q, x) *is implicit in* G iff x is parsable from q using G.

Given a grammar G, we define a function $\mathbf{remRedun}_G \in \mathcal{P}\, P_G \times \mathcal{P}\, P_G \to \mathcal{P}\, P_G$ by well-founded recursion on the size of its second argument.

For $U, V \subseteq P_G$, $\mathbf{remRedun}(U, V)$ proceeds as follows:

- If $V = \emptyset$, then it returns U.
- Otherwise, let v be the greatest element of $\{\, (q, x) \in V \mid$ there are no $p \in \mathbf{Sym}$ and $y \in \mathbf{Str}$ such that $(p, y) \in V$ and $|y| > |x| \,\}$, and $V' = V - \{v\}$. If v is implicit in $G/(U \cup V')$, then **remRedun** returns the result of evaluating $\mathbf{remRedun}(U, V')$. Otherwise, it returns the result of evaluating $\mathbf{remRedun}(U \cup \{v\}, V')$.

Our algorithm for removing redundant productions of a grammar G returns $G/(\mathbf{remRedun}_G(\emptyset, P_G))$.

Algorithm for Removing Redundant Productions

For example, if we run our algorithm for removing redundant productions on

$$\mathsf{A} \rightarrow \% \mid 0 \mid \mathsf{AA} \mid \mathsf{AAA},$$

we obtain

$$\mathsf{A} \rightarrow \% \mid 0 \mid \mathsf{AA}.$$

Simplification Algorithm

Our simplification algorithm for grammars proceeds as follows, given a grammar G.

- First, it determines which variables of G are generating. If s_G isn't one of these variables, then it returns the grammar with variable s_G and no productions.
- Next, it turns G into a grammar G' by deleting all non-generating variables, and deleting all productions involving such variables.
- Then, it determines which variables of G' are reachable.
- Next, it turns G' into a grammar G'' by deleting all non-reachable variables, and deleting all productions involving such variables.
- Finally, it removes redundant productions from G''.

Simplification Example

Suppose G, once again, is the grammar

$$\begin{aligned} \mathsf{A} &\to \mathsf{BB1}, \\ \mathsf{B} &\to \mathsf{0} \mid \mathsf{A} \mid \mathsf{CD}, \\ \mathsf{C} &\to \mathsf{12}, \\ \mathsf{D} &\to \mathsf{1D2}. \end{aligned}$$

Here is what happens if we apply our simplification algorithm to G.

First, we determine which variables are generating. Clearly B and C are. And, since B is, it follows that A is, because of the production $\mathsf{A} \to \mathsf{BB1}$. (If this production had been $\mathsf{A} \to \mathsf{BD1}$, we wouldn't have added A to our set.)

Simplification Example (Cont.)

Thus, we form G' from G by deleting the variable D, yielding the grammar

$$\begin{aligned} \mathsf{A} &\to \mathsf{BB1}, \\ \mathsf{B} &\to \mathsf{0} \mid \mathsf{A}, \\ \mathsf{C} &\to \mathsf{12}. \end{aligned}$$

Next, we determine which variables of G' are reachable. Clearly A is, and thus B is, because of the production $\mathsf{A} \to \mathsf{BB1}$.

Note that, if we carried out the two stages of our simplification algorithm in the other order, then C and its production would never be deleted.

Simplification Example (Cont.)

Next, we form G'' from G' by deleting the variable C, yielding the grammar

$$\begin{aligned} \mathsf{A} &\to \mathsf{BB1}, \\ \mathsf{B} &\to \mathsf{0} \mid \mathsf{A}. \end{aligned}$$

Finally, we would remove redundant productions from G''. But G'' has no redundant productions, and so we are done.

Simplification Function

We define a function **simplify** $\in$ **Gram** $\to$ **Gram** by: for all $G \in$ **Gram**, **simplify** G is the result of running the above algorithm on G.

Theorem 4.4.2

For all $G \in$ **Gram***:*

(1) **simplify** G *is simplified;*

(2) **simplify** $G \approx G$*; and*

(3) **alphabet**(**simplify** G) $=$ **alphabet**($L(G)$) $\subseteq$ **alphabet** G.

Testing Whether $L(G) = \emptyset$

Our simplification algorithm gives us an algorithm for testing whether the language generated by a grammar G is empty. We first simplify G, calling the result H. We then test whether $P_H = \emptyset$. If the answer is "yes", clearly $L(G) = L(H) = \emptyset$. And if the answer is "no", then s_H is useful, and so H (and thus G) generates at least one string.

Simplification in Forlan

The Forlan module `Gram` defines the functions

```
val simplify   : gram -> gram
val simplified : gram -> bool
```

Suppose gram of type gram is bound to the grammar

$$\begin{aligned} \mathsf{A} &\to \mathsf{BB1}, \\ \mathsf{B} &\to \mathsf{0} \mid \mathsf{A} \mid \mathsf{CD}, \\ \mathsf{C} &\to \mathsf{12}, \\ \mathsf{D} &\to \mathsf{1D2}. \end{aligned}$$

We can simplify our grammar as follows:

```
- val gram' = Gram.simplify gram;
val gram' = - : gram
- Gram.output("", gram');
{variables} A, B {start variable} A
{productions} A -> BB1; B -> 0 | A
val it = () : unit
```

Forlan Examples

Suppose `gram''` of type `gram` is bound to the grammar

$$\mathsf{A} \rightarrow \% \mid \mathsf{0} \mid \mathsf{AA} \mid \mathsf{AAA} \mid \mathsf{AAAA}.$$

We can simplify our grammar as follows:

```
- val gram''' = Gram.simplify gram'';
val gram''' = - : gram
- Gram.output("", gram''');
{variables} A {start variable} A
{productions} A -> % | 0 | AA
val it = () : unit
```

Hand-simplification Operations

Given a simplified grammar G, there are often ways we can hand-simplify the grammar further. Below are two examples.

Suppose G has a variable q that is not s_G, and where no production having q as its left-hand side is *self-recursive*, i.e., has q as one of the symbols of its right-hand side. Let $x_1, \ldots, x_n$ be the right-hand sides of all of q's productions.

Then we can form an equivalent grammar G' by deleting q and its productions from G, and transforming each remaining production $p \to y$ of G into all the productions from p that can be formed by substituting for each occurrence of q in y some choice of x_i.

We refer to this operation as *eliminating* q *from* G.

Hand-simplification Operations

Suppose there is exactly one production of G involving s_G, where that production has the form $s_G \to q$, for some variable q of G.

Then we can form an equivalent grammar G' by deleting s_G and $s_G \to q$ from G, and making q be the start variable of G'.

We refer to this operation as *restarting* G.

Hand-simplification Operations

The Forlan module `Gram` has functions corresponding to these two operations:

```
val eliminateVariable : gram * sym -> gram
val restart           : gram -> gram
```

Both begin by simplifying the supplied grammar.

For instance, suppose `gram` is the grammar

$$
\begin{aligned}
\mathsf{A} &\to \mathsf{B}, \\
\mathsf{B} &\to \mathsf{0} \mid \mathsf{C3C}, \\
\mathsf{C} &\to \mathsf{1B2} \mid \mathsf{2B1}.
\end{aligned}
$$

Hand-simplification Operations

Then we can proceed as follows:

```
- val gram' =
=       Gram.eliminateVariable
=       (gram, Sym.fromString "C");
val gram' = - : gram
- Gram.output("", gram');
{variables} A, B {start variable} A
{productions}
A -> B; B -> 0 | 1B231B2 | 1B232B1 | 2B131B2 | 2B132B1
val it = () : unit
- val gram'' = Gram.restart gram;
val gram'' = - : gram
- Gram.output("", gram'');
{variables} B, C {start variable} B
{productions} B -> 0 | C3C; C -> 1B2 | 2B1
val it = () : unit
```