4.10: The Pumping Lemma for Context-free Languages

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Question: is L context-free? I.e., is there a grammar that generates L?

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Question: is L context-free? I.e., is there a grammar that generates L?

Answer: No. Intuitively, although it's easy to keep the 0's and 1's matched, or to keep the 1's and 2's matched, or to keep the 0's and 2's matched, there is no way to keep all three symbols matched simultaneously.

Introduction

In this section, we will study the pumping lemma for context-free languages, which can be used to show that many languages are not context-free.

We will use the pumping lemma to prove that L is not context-free, and then we will prove the lemma.

Building on this result, we'll be able to show that the context-free languages are not closed under intersection, complementation or set-difference.

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(1) $|vwx| \leq n$;

(2) $vx \neq \%$; *and*

(3) $uv^i wx^i y \in L$, *for all* $i \in \mathbb{N}$.

Example Use of Pumping Lemma

Before proving the pumping lemma, let's see how it can be used to show that $L = \{\, 0^n 1^n 2^n \mid n \in \mathbb{N} \,\}$ is not context-free. Suppose, toward a contradiction that L is context-free. Thus there is an $n \in \mathbb{N} - \{0\}$ with the property of the lemma. Let $z =$

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(1) $|vwx| \leq n$;

(2) $vx \neq \%$; and

(3) $uv^i wx^i y \in L$, for all $i \in \mathbb{N}$.

Since $0^n 1^n 2^n = z = uvwxy$, (1) tells us that vwx

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(1) $|vwx| \leq n$;

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(3) $uv^i wx^i y \in L$, for all $i \in \mathbb{N}$.

Since $0^n 1^n 2^n = z = uvwxy$, (1) tells us that vwx doesn't contain both a 0 and a 2. Thus, vwx has no 0's or vwx has no 2's, so that there are two cases to consider.

Example

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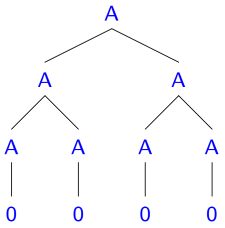
Since we obtained a contradiction in both cases, we have an overall contradiction. Thus L is not context-free.

A Fact About CNF Grammars

When we prove the pumping lemma for context-free languages, we will make use of a fact about grammars in Chomsky Normal Form.

Suppose G is a grammar in CNF and that $w \in (\textbf{alphabet}\ G)^*$ is the yield of a valid parse tree pt for G whose root label is a variable.

For instance, if G is the grammar with variable A and productions $\mathsf{A} \rightarrow \mathsf{AA}$ and $\mathsf{A} \rightarrow \mathsf{0}$, then w could be $\mathsf{0000}$ and pt could be the following tree of height 3:



CNF Fact

Generalizing from this example, we can see that if pt has height 3, $|w|$ will never be greater than $4 = 2^2 = 2^{3-1}$.

Question: how can we express an upper bound for $|w|$ as a function of the height k of pt?

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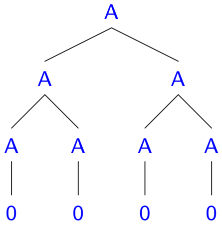
Maximal Length Paths through Parse Trees

A maximal length path through a tree of height k whose root label is a variable will have length k and will pass through exactly k variables.

E.g., the path $[1, 2, 1]$ through



(which goes to the left child, then to the right child, then to the only child (the second leaf from the left)) has length 3 and visits 3 variables.

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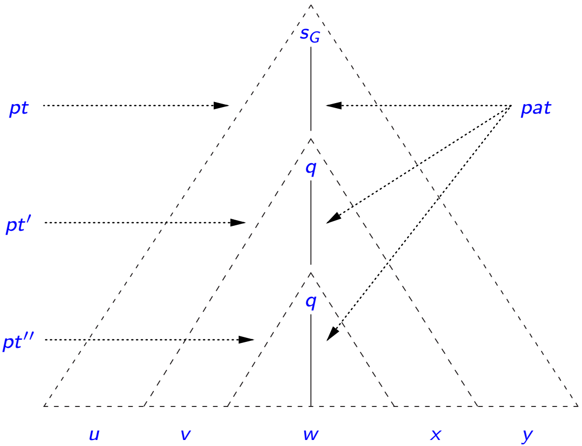
Proof (cont.). The rest of the proof can be visualized using the diagram



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Let pt' and pt'' be the subtrees of pt at positions pat' and pat'', i.e., the positions of the upper and lower occurrences of q, respectively.

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Consider the tree formed from pt by replacing the subtree at position pat' by q. This tree has yield

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Let w be the yield of pt''. Thus vwx is the yield of pt', so that $z = uvwxy$ is the yield of pt. Because the height of pt' is at most $k+1$, our fact about valid parse trees of grammars in CNF, tells us that $|vwx| \leq$

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Proof

Proof (cont.). Because G is in CNF, pt', which has q as its root label, has two children. The child whose root node isn't visited by pat''' will have a non-empty yield, and this yield will be a prefix of , if this child is the left child, and will be a suffix of , if this child is the right child.

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It remains to show part (3), i.e., that $uv^iwx^iy \in L(G) \subseteq L$, for all $i \in \mathbb{N}$. We define a valid parse tree pt_i for G, with root label q and yield v^iwx^i, by recursion on $i \in \mathbb{N}$. We let pt_0 be pt''. Then, if $i \in \mathbb{N}$, we form pt_{i+1} from pt' by replacing the subtree at position pat''' by pt_i.

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Suppose $i \in \mathbb{N}$. Then the parse tree formed from pt by replacing the subtree at position pat' by pt_i is valid for G, has root label s_G, and has yield $uv^i wx^i y$, showing that $uv^i wx^i y \in L(G)$. □

Forlan Implementation of Pumping Lemma

The textbook describes the implementation in Forlan of the idea behind the Pumping Lemma.

Consequences of Pumping Lemma

Suppose

$$\begin{aligned} L &= \{\, 0^n 1^n 2^n \mid n \in \mathbb{N} \,\}, \\ A &= \{\, 0^n 1^n 2^m \mid n, m \in \mathbb{N} \,\}, \text{ and} \\ B &= \{\, 0^n 1^m 2^m \mid n, m \in \mathbb{N} \,\}. \end{aligned}$$

Of course, L is not context-free.

Question: are A and B context-free?

Answer:

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Answer: yes, it's easy to find grammars that generate them.

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Question: is $A \cap B$ context-free?

Answer: no—$A \cap B = L$, and L is not context-free.

Thus the context-free languages are not closed under intersection.

Consequences

Question: is $\{0, 1, 2\}^* - A$ context-free?

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Consequences

Question: is $\{0,1,2\}^* - A$ context-free?

Answer: yes, since this language is the union of the context-free languages

$$\{0,1,2\}^* - \{0\}^*\{1\}^*\{2\}^*$$

and

$$\{\, 0^{n_1} 1^{n_2} 2^m \mid n_1, n_2, m \in \mathbb{N} \text{ and } n_1 \neq n_2 \,\},$$

(the first of these languages is regular), and the context-free languages are closed under union.

Similarly, we have that $\{0,1,2\}^* - B$ is context-free.

Consequences

Let

$$C = (\{0,1,2\}^* - A) \cup (\{0,1,2\}^* - B).$$

Thus C is a context-free subset of $\{0,1,2\}^*$. Since $A, B \subseteq \{0,1,2\}^*$, it is easy to show that

$$\begin{aligned} A \cap B &= \{0,1,2\}^* - ((\{0,1,2\}^* - A) \cup (\{0,1,2\}^* - B)) \\ &= \{0,1,2\}^* - C. \end{aligned}$$

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Thus

$$\{0,1,2\}^* - C = A \cap B = L$$

is not context-free. Thus the context-free languages are not closed under complementation or set difference.