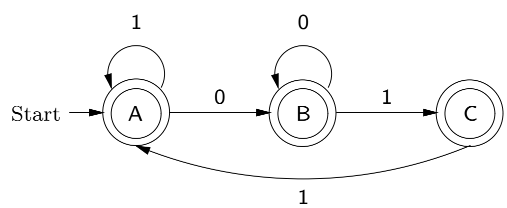
CS 516—Software Foundations via Formal Languages—Spring 2022

Problem Set 4

Model Answers

Problem 1

(a) The finite automaton N is



(b) First, we put the expression of N in Forlan's syntax

```
{states} A, B, C {start state} A {accepting states} A, B, C
{transitions}
A, 0 -> B; A, 1 -> A;
B, 0 -> B; B, 1 -> C;
C, 1 -> A
```

in the file `ps4-p1-fa` (see the course website), and load this file into Forlan, calling the result `fa`:

```
- val fa = FA.input "ps4-p1-fa";
val fa = - : fa
```

Next we load the file `ps4-p1.sml`

```
(* val inX : str -> bool

   tests whether a string over the alphabet {0, 1} is in X *)

fun inX x =
      Set.all
      (fn y => not(Str.equal(y, Str.fromString "010")))
      (StrSet.substrings x);

(* val upto : int -> str set

   if n >= 0, then upto n returns all strings over alphabet {0, 1} of
   length no more than n *)

fun upto 0 : str set = Set.sing nil
```

```
  | upto n           =
      let val xs = upto(n - 1)
          val ys = Set.filter (fn x => length x = n - 1) xs
      in StrSet.union
         (xs, StrSet.concat(StrSet.fromString "0, 1", ys))
      end;

(* val partition : int -> str set * str set

   if n >= 0, then partition n returns (xs, ys) where:

   xs is all elements of upto n that are in X; and

   ys is all elements of upto n that are not in X *)

fun partition n = Set.partition inX (upto n);

(* val test = fn : int -> fa -> str option * str option

   if n >= 0, then test n returns a function f such that, for all FAs
   fa, f fa returns a pair (xOpt, yOpt) such that:

     If there is an element of {0, 1}* of length no more than n that
     is in X but is not accepted by fa, then xOpt = SOME x for some
     such x; otherwise, xOpt = NONE.

     If there is an element of {0, 1}* of length no more than n that
     is not in X but is accepted by fa, then yOpt = SOME y for some
     such y; otherwise, yOpt = NONE. *)

fun test n =
      let val (goods, bads) = partition n
      in fn fa =>
              let val accepted      = FA.accepted fa
                  val goodNotAccOpt = Set.position (not o accepted) goods
                  val badAccOpt     = Set.position accepted bads
              in ((case goodNotAccOpt of
                        NONE   => NONE
                      | SOME i => SOME(ListAux.sub(Set.toList goods, i))),
                  (case badAccOpt of
                        NONE   => NONE
                      | SOME i => SOME(ListAux.sub(Set.toList bads, i))))
              end
      end;
```

(see the course website) defining the function `test` into Forlan:

```
- use "ps4-p1.sml";
[opening ps4-p1.sml]
```

```
val inX = fn : str -> bool
val upto = fn : int -> str set
val partition = fn : int -> str set * str set
val test = fn : int -> fa -> str option * str option
val it = () : unit
```

Finally, we apply `test` to arguments `10` and `fa`:

```
- test 10 fa;
val it = (NONE,NONE) : str option * str option
```

Problem 2

(a) First, we load the file `ps4-p2-fa` (see the course website) containing the expression

```
{states} A, B, C, D {start state} A {accepting states} B, C, D
{transitions}
A, % -> B | C | D;
B, 0 -> C;
C, 0 -> D; C, 1 -> B;
D, 1 -> C
```

of M in Forlan's syntax into Forlan, calling the result `fa`:

```
- val fa = FA.input "ps4-p2-fa";
val fa = - : fa
```

Next, we define a function `accPr` that finds and prints a labeled path in `fa` explaining why a Forlan string expressed as an SML string is accepted:

```
- fun accPr s =
=       LP.output("", FA.findAcceptingLP fa (Str.fromString s));
val accPr = fn : string -> unit
```

Finally, we use this function to find and display the required labeled paths:

```
- accPr "0010110";
A, % => B, 0 => C, 0 => D, 1 => C, 0 => D, 1 => C, 1 => B, 0 => C
val it = () : unit
- accPr "1001101";
A, % => C, 1 => B, 0 => C, 0 => D, 1 => C, 1 => B, 0 => C, 1 => B
val it = () : unit
- accPr "1011001";
A, % => D, 1 => C, 0 => D, 1 => C, 1 => B, 0 => C, 0 => D, 1 => C
val it = () : unit
```

(b) Continuing our Forlan session, we first load the file `ps4-p2.sml`

```
fun accLen n =
      Set.filter
      (FA.accepted fa)
      (StrSet.power(StrSet.fromString "0,1", n));
```

(see the course website) defining the function `accLen` into Forlan:

```
- use "ps4-p2.sml";
[opening ps4-p2.sml]
val accLen = fn : int -> str set
val it = () : unit
```

Then we apply it to 10, calling the resulting set of labeled paths `lps`, compute the size of `lps`, and display its elements:

```
- val lps = accLen 10;
val lps = - : str set
- Set.size lps;
val it = 94 : int
- StrSet.output("", lps);
0010101010, 0010101011, 0010101100, 0010101101, 0010110010, 0010110011,
0010110100, 0010110101, 0011001010, 0011001011, 0011001100, 0011001101,
0011010010, 0011010011, 0011010100, 0011010101, 0100101010, 0100101011,
0100101100, 0100101101, 0100110010, 0100110011, 0100110100, 0100110101,
0101001010, 0101001011, 0101001100, 0101001101, 0101010010, 0101010011,
0101010100, 0101010101, 0101010110, 0101011001, 0101011010, 0101100101,
0101100110, 0101101001, 0101101010, 0110010101, 0110010110, 0110011001,
0110011010, 0110100101, 0110100110, 0110101001, 0110101010, 1001010101,
1001010110, 1001011001, 1001011010, 1001100101, 1001100110, 1001101001,
1001101010, 1010010101, 1010010110, 1010011001, 1010011010, 1010100101,
1010100110, 1010101001, 1010101010, 1010101011, 1010101100, 1010101101,
1010110010, 1010110011, 1010110100, 1010110101, 1011001010, 1011001011,
1011001100, 1011001101, 1011010010, 1011010011, 1011010100, 1011010101,
1100101010, 1100101011, 1100101100, 1100101101, 1100110010, 1100110011,
1100110100, 1100110101, 1101001010, 1101001011, 1101001100, 1101001101,
1101010010, 1101010011, 1101010100, 1101010101
val it = () : unit
```

Problem 3

Define a function **dsfxs** (for "diffs of suffixes") from $\{0,1\}^*$ to $\mathcal{P}\,\mathbb{Z}$ by: for all $w \in \{0,1\}^*$,

$$\mathbf{dsfxs}\,w = \{\, \mathbf{diff}\, v \mid v \text{ is a suffix of } w \,\}.$$

From the definitions of X and **dsfxs** and the fact that suffixes are substrings, we have that, if $w \in X$, then $\mathbf{dsfxs}\,w \subseteq \{-2,-1,0,1,2\}$. It turns out, though, that we can characterize membership in X using **dsfxs**.

Lemma PS4.3.1
For all $w \in X$ *and* $n, m \in \mathbf{dsfxs}\,w$, $-2 \leq m - n \leq 2$.

Proof. Suppose $w \in X$ and $n, m \in \mathbf{dsfxs}\,w$, so that there are suffixes u and v of w such that $n = \mathbf{diff}\,u$ and $m = \mathbf{diff}\,v$. Because u and v are suffixes of w, one must be a suffix of the other, and so there are two cases to consider.

- Suppose u is a suffix of v. Thus $v = zu$ for some $z \in \{0,1\}^*$, and thus z is a substring of w. Hence $m = \textbf{diff}\, v = \textbf{diff}\, z + \textbf{diff}\, u = \textbf{diff}\, z + n$, so that $m - n = \textbf{diff}\, z$. Because z is a substring of $w \in X$, we have that $-2 \leq \textbf{diff}\, z \leq 2$, and thus $-2 \leq m - n \leq 2$.
- Suppose v is a suffix of u. Thus $u = zv$ for some $z \in \{0,1\}^*$, and thus z is a substring of w. Hence $n = \textbf{diff}\, u = \textbf{diff}\, z + \textbf{diff}\, v = \textbf{diff}\, z + m$, so that $n - m = \textbf{diff}\, z$. Because z is a substring of $w \in X$, we have that $-2 \leq \textbf{diff}\, z \leq 2$, and thus $-2 \leq n - m \leq 2$. Since $-2 \leq n - m$, we have that $m - n = -(n - m) \leq -(-2) = 2$. And since $n - m \leq 2$, we have that $-2 \leq -(n - m) = m - n$. Thus $-2 \leq m - n \leq 2$.

□

Lemma PS4.3.2
For all $w \in X$, either $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1, 0, 1\}$ or $\textbf{dsfxs}\, w \subseteq \{0, 1, 2\}$.

Proof. Suppose $w \in X$. Thus $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0, 1, 2\}$. Because % is a suffix of w, we have that $0 = \textbf{diff}\, \% \in \textbf{dsfxs}\, w$. There are two cases to consider.

- Suppose $-2 \in \textbf{dsfxs}\, w$. Lemma PS4.3.1 tells us that neither 1 nor 2 are elements of $\textbf{dsfxs}\, w$, since $-2 - 1 = -3$ and $-2 - 2 = -4$ are both < -2. Thus $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0\}$, so that either $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1, 0, 1\}$ or $\textbf{dsfxs}\, w \subseteq \{0, 1, 2\}$.
- Suppose $-2 \notin \textbf{dsfxs}\, w$. Then $\textbf{dsfxs}\, w \subseteq \{-1, 0, 1, 2\}$. There are two subcases to consider.
  - Suppose $2 \in \textbf{dsfxs}\, w$. Then Lemma PS4.3.1 tells us that -1 is not an element of $\textbf{dsfxs}\, w$, since $2 - (-1) = 3$ is > 2. Thus $\textbf{dsfxs}\, w \subseteq \{0, 1, 2\}$, so that either $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1, 0, 1\}$ or $\textbf{dsfxs}\, w \subseteq \{0, 1, 2\}$.
  - Suppose $2 \notin \textbf{dsfxs}\, w$. Then $\textbf{dsfxs}\, w \subseteq \{-1, 0, -1\}$, so that either $\textbf{dsfxs}\, w \subseteq \{-2, -1, 0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1, 0, 1\}$ or $\textbf{dsfxs}\, w \subseteq \{0, 1, 2\}$.

□

Lemma PS4.3.3
For all $w \in \{0,1\}^*$ and $n \in \{-2, -1, 0\}$, if $\textbf{dsfxs}\, w \subseteq \{n, n+1, n+2\}$, then $w \in X$.

Proof. Suppose $w \in \{0,1\}^*$, $n \in \{-2, -1, 0\}$ and $\textbf{dsfxs}\, w \subseteq \{n, n+1, n+2\}$. To show that $w \in X$, suppose v is a substring of w. Thus $w = xvy$ for some $x, y \in \{0,1\}^*$. We must show that $-2 \leq \textbf{diff}\, v \leq 2$. Because y is a suffix of w, $\textbf{diff}\, y \in \textbf{dsfxs}\, w$, and thus $n \leq \textbf{diff}\, y \leq n + 2$. Because vy is a suffix of w, $\textbf{diff}(vy) \in \textbf{dsfxs}\, w$, and thus $n \leq \textbf{diff}(vy) \leq n + 2$. And since $\textbf{diff}(vy) = \textbf{diff}\, v + \textbf{diff}\, y = \textbf{diff}\, y + \textbf{diff}\, v$, it follows that $n \leq \textbf{diff}\, y + \textbf{diff}\, v \leq n + 2$.

Suppose, toward a contradiction, that $-2 \leq \textbf{diff}\, v \leq 2$ is false. Thus there are two cases to consider.

- Suppose $\textbf{diff}\, v \leq -3$. Because $\textbf{diff}\, y \leq n+2$, it follows that $n \leq \textbf{diff}\, y + \textbf{diff}\, v \leq (n+2) + -3 = n - 1$, so that $n \leq n - 1$—contradiction.
- Suppose $3 \leq \textbf{diff}\, v$. Because $n \leq \textbf{diff}\, y$, it follows that $n + 3 \leq \textbf{diff}\, y + \textbf{diff}\, v \leq n + 2$, so that $3 \leq 2$—contradiction.

Because we obtained a contradiction in both cases, we have an overall contradiction. Thus $-2 \leq \textbf{diff}\, v \leq 2$, completing the proof that $w \in X$. $\square$

Lemma PS4.3.4
For all $w \in \{0,1\}^*$, if either $\textbf{dsfxs}\, w \subseteq \{-2,-1,0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1,0,1\}$ or $\textbf{dsfxs}\, w \subseteq \{0,1,2\}$, then $w \in X$.

Proof. Suppose $w \in \{0,1\}^*$ and assume that either $\textbf{dsfxs}\, w \subseteq \{-2,-1,0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1,0,1\}$ or $\textbf{dsfxs}\, w \subseteq \{0,1,2\}$. There are three case to consider.

- Suppose $\textbf{dsfxs}\, w \subseteq \{-2,-1,0\}$. Because $-2 \in \{-2,-1,0\}$ and $\textbf{dsfxs}\, w \subseteq \{-2,-1,0\} = \{-2,(-2)+1,(-2)+2\}$, Lemma PS4.3.3 tells us that $w \in X$.
- Suppose $\textbf{dsfxs}\, w \subseteq \{-1,0,1\}$. Because $-1 \in \{-2,-1,0\}$ and $\textbf{dsfxs}\, w \subseteq \{-1,0,1\} = \{-1,(-1)+1,(-1)+2\}$, Lemma PS4.3.3 tells us that $w \in X$.
- Suppose $\textbf{dsfxs}\, w \subseteq \{0,1,2\}$. Because $0 \in \{-2,-1,0\}$ and $\textbf{dsfxs}\, w \subseteq \{0,1,2\} = \{0,0{+}1,0{+}2\}$, Lemma PS4.3.3 tells us that $w \in X$.

$\square$

For $-2 \leq n \leq 0 \leq m \leq 2$, define

$$Y^{n,m} = \{\, w \in \{0,1\}^* \mid \textbf{dsfxs}\, w \subseteq \{n,\ldots,m\}\,\}.$$

Thus it is easy to show that:

- if v is a suffix of $w \in Y^{n,m}$, then $n \leq \textbf{diff}\, v \leq m$;
- if $w \in \{0,1\}^*$ and, for all suffixes v of w, $n \leq \textbf{diff}\, v \leq m$, then $w \in Y^{n,m}$;
- $\% \in Y^{n,m}$.

The basis of the proof that $L(M) = X$ is the following lemma:

Lemma PS4.3.5
(1) $X = Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$.

(2) $Y^{0,2} = \{\%\} \cup Y^{-1,1}\{1\}$.

(3) $Y^{-1,1} = \{\%\} \cup Y^{0,2}\{0\} \cup Y^{-2,0}\{1\}$.

(4) $Y^{-2,0} = \{\%\} \cup Y^{-1,1}\{0\}$.

Proof.

(1) We show that $X \subseteq Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0} \subseteq X$.

- To show $X \subseteq Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$, suppose $w \in X$. By Lemma PS4.3.2, we have that either $\textbf{dsfxs}\, w \subseteq \{-2,-1,0\}$ or $\textbf{dsfxs}\, w \subseteq \{-1,0,1\}$ or $\textbf{dsfxs}\, w \subseteq \{0,1,2\}$. Thus there are three cases to consider.

- Suppose $\mathbf{dsfxs}\, w \subseteq \{-2,-1,0\}$. Thus $w \in Y^{-2,0} \subseteq Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$.
- Suppose $\mathbf{dsfxs}\, w \subseteq \{-1,0,1\}$. Thus $w \in Y^{-1,1} \subseteq Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$.
- Suppose $\mathbf{dsfxs}\, w \subseteq \{0,1,2\}$. Thus $w \in Y^{0,2} \subseteq Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$.

- To show $Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0} \subseteq X$, suppose $w \in Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0}$. There are three cases to consider.
  - Suppose $w \in Y^{0,2}$, so that $\mathbf{dsfxs}\, w \subseteq \{0,1,2\}$. Thus $w \in X$, by Lemma PS4.3.4.
  - Suppose $w \in Y^{-1,1}$, so that $\mathbf{dsfxs}\, w \subseteq \{-1,0,1\}$. Thus $w \in X$, by Lemma PS4.3.4.
  - Suppose $w \in Y^{-2,0}$, so that $\mathbf{dsfxs}\, w \subseteq \{-2,-1,0\}$. Thus $w \in X$, by Lemma PS4.3.4.

(2) We show that $Y^{0,2} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{1}\} \subseteq Y^{0,2}$.

- To show that $Y^{0,2} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{1}\}$, suppose $w \in Y^{0,2}$. If $w = \%$, then $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{1}\}$. So, suppose $w \neq \%$. Then $w = xa$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$ and $a \in \{\mathsf{0},\mathsf{1}\}$. We cannot have $a = \mathsf{0}$, as then $-1 \in \mathbf{dsfxs}\, w$ (contradicting $w \in Y^{0,2}$). Thus $a = \mathsf{1}$, so that $w = x\mathsf{1}$. To see that $x \in Y^{-1,1}$, suppose v is a suffix of x. Because $v\mathsf{1}$ is a suffix of $w \in Y^{0,2}$, we have that $0 \leq \mathbf{diff}(v\mathsf{1}) \leq 2$. But $\mathbf{diff}(v\mathsf{1}) = \mathbf{diff}\, v + 1$, and thus $-1 \leq \mathbf{diff}\, v \leq 1$. Thus $w = x\mathsf{1} \in Y^{-1,1}\{\mathsf{1}\} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{1}\}$.
- To show that $\{\%\} \cup Y^{-1,1}\{\mathsf{1}\} \subseteq Y^{0,2}$, suppose $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{1}\}$. If $w \in \{\%\}$, then $w \in Y^{0,2}$. Otherwise, we have that $w \in Y^{-1,1}\{\mathsf{1}\}$, so that $w = x\mathsf{1}$, for some $x \in Y^{-1,1}$. To see that $w \in Y^{0,2}$, suppose v is a suffix of $w = x\mathsf{1}$. We must show that $0 \leq \mathbf{diff}\, v \leq 2$. If $v = \%$, then this is true. Otherwise $v = u\mathsf{1}$ for some suffix u of x. Because $x \in Y^{-1,1}$, we have that $-1 \leq \mathbf{diff}\, u \leq 1$. Thus $0 \leq \mathbf{diff}\, v \leq 2$.

(3) We show that $Y^{-1,1} \subseteq \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\} \subseteq Y^{-1,1}$.

- To show that $Y^{-1,1} \subseteq \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$, suppose $w \in Y^{-1,1}$. If $w = \%$, then $w \in \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$. So, suppose $w \neq \%$. Then $w = xa$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$ and $a \in \{\mathsf{0},\mathsf{1}\}$. There are two cases to consider.
  - Suppose $a = \mathsf{0}$, so that $w = x\mathsf{0}$. To see that $x \in Y^{0,2}$, suppose v is a suffix of x. Because $v\mathsf{0}$ is a suffix of $w \in Y^{-1,1}$, we have that $-1 \leq \mathbf{diff}(v\mathsf{0}) \leq 1$. But $\mathbf{diff}(v\mathsf{0}) = \mathbf{diff}\, v + -1$, and thus $0 \leq \mathbf{diff}\, v \leq 2$. Thus $w = x\mathsf{0} \in Y^{0,2}\{\mathsf{0}\} \subseteq \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$.
  - Suppose $a = \mathsf{1}$, so that $w = x\mathsf{1}$. To see that $x \in Y^{-2,0}$, suppose v is a suffix of x. Because $v\mathsf{1}$ is a suffix of $w \in Y^{-1,1}$, we have that $-1 \leq \mathbf{diff}(v\mathsf{1}) \leq 1$. But $\mathbf{diff}(v\mathsf{1}) = \mathbf{diff}\, v + 1$, and thus $-2 \leq \mathbf{diff}\, v \leq 0$. Thus $w = x\mathsf{1} \in Y^{-2,0}\{\mathsf{1}\} \subseteq \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$.
- To show that $\{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\} \subseteq Y^{-1,1}$, suppose $w \in \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$. If $w \in \{\%\}$, then $w \in Y^{-1,1}$. Otherwise, there are two cases to consider.
  - Suppose $w \in Y^{0,2}\{\mathsf{0}\}$, so that $w = x\mathsf{0}$, for some $x \in Y^{0,2}$. To see that that $w \in Y^{-1,1}$, suppose v is a suffix of $w = x\mathsf{0}$. We must show that $-1 \leq \mathbf{diff}\, v \leq 1$. If $v = \%$, then this is true. Otherwise $v = u\mathsf{0}$ for some suffix u of x. Because $x \in Y^{0,2}$, we have that $0 \leq \mathbf{diff}\, u \leq 2$. Thus $-1 \leq \mathbf{diff}\, v \leq 1$.

– Suppose $w \in Y^{-2,0}\{\mathsf{1}\}$, so that $w = x\mathsf{1}$, for some $x \in Y^{-2,0}$. To see that $w \in Y^{-1,1}$, suppose v is a suffix of $w = x\mathsf{1}$. We must show that $-1 \leq \mathbf{diff}\, v \leq 1$. If $v = \%$, then this is true. Otherwise $v = u\mathsf{1}$ for some suffix u of x. Because $x \in Y^{-2,0}$, we have that $-2 \leq \mathbf{diff}\, u \leq 0$. Thus $-1 \leq \mathbf{diff}\, v \leq 1$.

(4) We show that $Y^{-2,0} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{0}\} \subseteq Y^{-2,0}$.

- To show that $Y^{-2,0} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{0}\}$, suppose $w \in Y^{-2,0}$. If $w = \%$, then $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{0}\}$. So, suppose $w \neq \%$. Then $w = xa$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$ and $a \in \{\mathsf{0},\mathsf{1}\}$. We cannot have $a = \mathsf{1}$, as then $1 \in \mathbf{dsfxs}\, w$ (contradicting $w \in Y^{-2,0}$). Thus $a = \mathsf{0}$, so that $w = x\mathsf{0}$. To see that $x \in Y^{-1,1}$, suppose v is a suffix of x. Because $v\mathsf{0}$ is a suffix of $w \in Y^{-2,0}$, we have that $-2 \leq \mathbf{diff}(v\mathsf{0}) \leq 0$. But $\mathbf{diff}(v\mathsf{0}) = \mathbf{diff}\, v + -1$, and thus $-1 \leq \mathbf{diff}\, v \leq 1$. Thus $w = x\mathsf{0} \in Y^{-1,1}\{\mathsf{0}\} \subseteq \{\%\} \cup Y^{-1,1}\{\mathsf{0}\}$.
- To show that $\{\%\} \cup Y^{-1,1}\{\mathsf{0}\} \subseteq Y^{-2,0}$, suppose $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{0}\}$. If $w \in \{\%\}$, then $w \in Y^{-2,0}$. Otherwise, we have that $w \in Y^{-1,1}\{\mathsf{0}\}$, so that $w = x\mathsf{0}$, for some $x \in Y^{-1,1}$. To see that $w \in Y^{-2,0}$, suppose v is a suffix of $w = x\mathsf{0}$. We must show that $-2 \leq \mathbf{diff}\, v \leq 0$. If $v = \%$, then this is true. Otherwise $v = u\mathsf{0}$ for some suffix u of x. Because $x \in Y^{-1,1}$, we have that $-1 \leq \mathbf{diff}\, u \leq 1$. Thus $-2 \leq \mathbf{diff}\, v \leq 0$.

□

In what follows, we will show that $\Lambda_{\mathsf{A}} = \{\%\}$, $\Lambda_{\mathsf{B}} = Y^{0,2}$, $\Lambda_{\mathsf{C}} = Y^{-1,1}$ and $\Lambda_{\mathsf{D}} = Y^{-2,0}$.

Lemma PS4.3.6

(A) For all $w \in \Lambda_{\mathsf{A}}$, $w \in \{\%\}$.

(B) For all $w \in \Lambda_{\mathsf{B}}$, $w \in Y^{0,2}$.

(C) For all $w \in \Lambda_{\mathsf{C}}$, $w \in Y^{-1,1}$.

(D) For all $w \in \Lambda_{\mathsf{D}}$, $w \in Y^{-2,0}$.

Proof. We proceed by induction on Λ. There are 8 (1 plus the number of transitions) parts to show.

(empty string) Clearly $\% \in \{\%\}$, as required.

$(\mathsf{A}, \% \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in \{\%\}$. We must show that $w\% \in Y^{0,2}$. And $w\% = \%\% = \% \in Y^{0,2}$.

$(\mathsf{A}, \% \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in \{\%\}$. We must show that $w\% \in Y^{-1,1}$. And $w\% = \%\% = \% \in Y^{-1,1}$.

$(\mathsf{A}, \% \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in \{\%\}$. We must show that $w\% \in Y^{-2,0}$. And $w\% = \%\% = \% \in Y^{-2,0}$.

$(\mathsf{B}, \mathsf{0} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Y^{0,2}$. We must show that $w\mathsf{0} \in Y^{-1,1}$. And $w\mathsf{0} \in Y^{0,2}\{\mathsf{0}\} \subseteq Y^{-1,1}$, by Lemma PS4.3.5(3).

($\mathsf{C}, 0 \to \mathsf{D}$) Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in Y^{-1,1}$. We must show that $w\mathsf{0} \in Y^{-2,0}$. And $w\mathsf{0} \in Y^{-1,1}\{\mathsf{0}\} \subseteq Y^{-2,0}$, by Lemma PS4.3.5(4).

($\mathsf{C}, 1 \to \mathsf{B}$) Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in Y^{-1,1}$. We must show that $w\mathsf{1} \in Y^{0,2}$. And $w\mathsf{1} \in Y^{-1,1}\{\mathsf{1}\} \subseteq Y^{0,2}$, by Lemma PS4.3.5(2).

($\mathsf{D}, 1 \to \mathsf{C}$) Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis: $w \in Y^{-2,0}$. We must show that $w\mathsf{1} \in Y^{-1,1}$. And $w\mathsf{1} \in Y^{-2,0}\{\mathsf{1}\} \subseteq Y^{-1,1}$, by Lemma PS4.3.5(3).

□

Lemma PS4.3.7
For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$:

(A) if $w \in \{\%\}$, then $w \in \Lambda_{\mathsf{A}}$;

(B) if $w \in Y^{0,2}$, then $w \in \Lambda_{\mathsf{B}}$;

(C) if $w \in Y^{-1,1}$, then $w \in \Lambda_{\mathsf{C}}$;

(D) if $w \in Y^{-2,0}$, then $w \in \Lambda_{\mathsf{D}}$.

Proof. We proceed by strong string induction. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$, if x is a proper substring of w, then

(A) if $x \in \{\%\}$, then $x \in \Lambda_{\mathsf{A}}$;

(B) if $x \in Y^{0,2}$, then $x \in \Lambda_{\mathsf{B}}$;

(C) if $x \in Y^{-1,1}$, then $x \in \Lambda_{\mathsf{C}}$;

(D) if $x \in Y^{-2,0}$, then $x \in \Lambda_{\mathsf{D}}$.

We must show that

(A) if $w \in \{\%\}$, then $w \in \Lambda_{\mathsf{A}}$;

(B) if $w \in Y^{0,2}$, then $w \in \Lambda_{\mathsf{B}}$;

(C) if $w \in Y^{-1,1}$, then $w \in \Lambda_{\mathsf{C}}$;

(D) if $w \in Y^{-2,0}$, then $w \in \Lambda_{\mathsf{D}}$.

There are four cases to consider.

(A) Suppose $w \in \{\%\}$. We must show that $w \in \Lambda_{\mathsf{A}}$. Because A is M's start state, $w = \% \in \Lambda_{\mathsf{A}}$.

(B) Suppose $w \in Y^{0,2}$. We must show that $w \in \Lambda_{\mathsf{B}}$. By Lemma PS4.3.5(2), we have that $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{1}\}$. Thus there are two subcases to consider.

- Suppose $w \in \{\%\}$. Because A is M's start state, we have $\% \in \Lambda_{\mathsf{A}}$. And since $(\mathsf{A}, \%, \mathsf{B}) \in T_M$, it follows that $w = \% = \%\% \in \Lambda_{\mathsf{B}}$.

- Suppose $w \in Y^{-1,1}\{\mathsf{1}\}$, so that $w = x\mathsf{1}$, for some $x \in Y^{-1,1}$. Because x is a proper substring of w, part (C) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{C}}$. Thus $w = x\mathsf{1} \in \Lambda_{\mathsf{B}}$, because of the transition $(\mathsf{C}, \mathsf{1}, \mathsf{B})$.

(C) Suppose $w \in Y^{-1,1}$. We must show that $w \in \Lambda_{\mathsf{C}}$. By Lemma PS4.3.5(3), we have that $w \in \{\%\} \cup Y^{0,2}\{\mathsf{0}\} \cup Y^{-2,0}\{\mathsf{1}\}$. Thus there are three subcases to consider.

- Suppose $w \in \{\%\}$. Because A is M's start state, we have $\% \in \Lambda_{\mathsf{A}}$. And since $(\mathsf{A}, \%, \mathsf{C}) \in T_M$, it follows that $w = \% = \%\% \in \Lambda_{\mathsf{C}}$.
- Suppose $w \in Y^{0,2}\{\mathsf{0}\}$, so that $w = x\mathsf{0}$, for some $x \in Y^{0,2}$. Because x is a proper substring of w, part (B) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{B}}$. Thus $w = x\mathsf{0} \in \Lambda_{\mathsf{C}}$, because of the transition $(\mathsf{B}, \mathsf{0}, \mathsf{C})$.
- Suppose $w \in Y^{-2,0}\{\mathsf{1}\}$, so that $w = x\mathsf{1}$, for some $x \in Y^{-2,0}$. Because x is a proper substring of w, part (D) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{D}}$. Thus $w = x\mathsf{1} \in \Lambda_{\mathsf{C}}$, because of the transition $(\mathsf{D}, \mathsf{1}, \mathsf{C})$.

(D) Suppose $w \in Y^{-2,0}$. We must show that $w \in \Lambda_{\mathsf{D}}$. By Lemma PS4.3.5(4), we have that $w \in \{\%\} \cup Y^{-1,1}\{\mathsf{0}\}$. Thus there are two subcases to consider.

- Suppose $w \in \{\%\}$. Because A is M's start state, we have $\% \in \Lambda_{\mathsf{A}}$. And since $(\mathsf{A}, \%, \mathsf{D}) \in T_M$, it follows that $w = \% = \%\% \in \Lambda_{\mathsf{D}}$.
- Suppose $w \in Y^{-1,1}\{\mathsf{0}\}$, so that $w = x\mathsf{0}$, for some $x \in Y^{-1,1}$. Because x is a proper substring of w, part (C) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{C}}$. Thus $w = x\mathsf{0} \in \Lambda_{\mathsf{D}}$, because of the transition $(\mathsf{C}, \mathsf{0}, \mathsf{D})$.

□

Lemma PS4.3.8

(A) $\Lambda_{\mathsf{A}} = \{\%\}$.

(B) $\Lambda_{\mathsf{B}} = Y^{0,2}$.

(C) $\Lambda_{\mathsf{C}} = Y^{-1,1}$.

(D) $\Lambda_{\mathsf{D}} = Y^{-2,0}$.

Proof. Follows by Lemmas PS4.3.6 and PS4.3.7. □

Lemma PS4.3.9
$L(M) = X$.

Proof. Because M's set of accepting states is $\{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$, it follows that $L(M) = \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}}$. And by Lemma PS4.3.8 and Lemma PS4.3.5(1), we have that $\Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} = Y^{0,2} \cup Y^{-1,1} \cup Y^{-2,0} = X$. Thus $L(M) = X$. □