CS 516—Software Foundations via Formal Languages—Spring 2025

Final Examination

Model Answers

Question 1

We can conclude that:

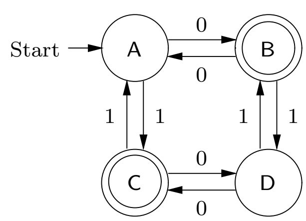
- if $j = 0$, then $j' = 0$ and $i + k = i' + k'$; and
- if $j \geq 1$, then $i = i'$, $j = j'$ and $k = k'$.

Question 2

$$\begin{aligned}\mathsf{A} &\to 0\mathsf{B}2 \mid 1\mathsf{C}2,\\ \mathsf{B} &\to 0\mathsf{B}2 \mid \mathsf{C},\\ \mathsf{C} &\to 1\mathsf{C}2 \mid \mathsf{C}2 \mid 2\end{aligned}$$

Question 3

M is



First, we show by induction on Λ that:

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X_{\mathsf{ee}}$;

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in X_{\mathsf{oe}}$;

(C) for all $w \in \Lambda_{\mathsf{C}}$, $w \in X_{\mathsf{eo}}$;

(D) for all $w \in \Lambda_{\mathsf{D}}$, $w \in X_{\mathsf{oo}}$.

There are nine (one plus the number of transitions) parts to show:

(empty string) We must show that $\% \in X_{\mathsf{ee}}$, and this follows by fact (1).

$(\mathsf{A}, \mathsf{0} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{ee}}$. Then $w\mathsf{0} \in X_{\mathsf{ee}}\{\mathsf{0}\} \subseteq X_{\mathsf{oe}}$, by fact (2).

$(\mathsf{A}, \mathsf{1} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{ee}}$. Then $w\mathsf{1} \in X_{\mathsf{ee}}\{\mathsf{1}\} \subseteq X_{\mathsf{eo}}$, by fact (3).

$(\mathsf{B}, \mathsf{0} \to \mathsf{A})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{oe}}$. Then $w\mathsf{0} \in X_{\mathsf{oe}}\{\mathsf{0}\} \subseteq X_{\mathsf{ee}}$, by fact (6).

$(\mathsf{B}, \mathsf{1} \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{oe}}$. Then $w\mathsf{1} \in X_{\mathsf{oe}}\{\mathsf{1}\} \subseteq X_{\mathsf{oo}}$, by fact (7).

$(\mathsf{C}, \mathsf{0} \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{eo}}$. Then $w\mathsf{0} \in X_{\mathsf{eo}}\{\mathsf{0}\} \subseteq X_{\mathsf{oo}}$, by fact (4).

$(\mathsf{C}, \mathsf{1} \to \mathsf{A})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{eo}}$. Then $w\mathsf{1} \in X_{\mathsf{eo}}\{\mathsf{1}\} \subseteq X_{\mathsf{ee}}$, by fact (5).

$(\mathsf{D}, \mathsf{0} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{oo}}$. Then $w\mathsf{0} \in X_{\mathsf{oo}}\{\mathsf{0}\} \subseteq X_{\mathsf{eo}}$, by fact (8).

$(\mathsf{D}, \mathsf{1} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis, $w \in X_{\mathsf{oo}}$. Then $w\mathsf{1} \in X_{\mathsf{oo}}\{\mathsf{1}\} \subseteq X_{\mathsf{oe}}$, by fact (9).

Now we use the result of our induction on Λ to show that $L(M) = X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$.

$(L(M) \subseteq X_{\mathsf{eo}} \cup X_{\mathsf{oe}})$ Suppose $w \in L(M)$. Because $A_M = \{\mathsf{B}, \mathsf{C}\}$, we have that $w \in L(M) = \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}}$. Thus there are two cases to consider:

- Suppose $w \in \Lambda_{\mathsf{B}}$. By part (B) of our induction on Λ, we have $w \in X_{\mathsf{oe}} \subseteq X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$.
- Suppose $w \in \Lambda_{\mathsf{C}}$. By part (C) of our induction on Λ, we have $w \in X_{\mathsf{eo}} \subseteq X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$.

$(X_{\mathsf{eo}} \cup X_{\mathsf{oe}} \subseteq L(M))$ Suppose $w \in X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$. Since $X_{\mathsf{eo}} \cup X_{\mathsf{oe}} \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Because $w \notin L(M) = \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}}$, and $w \in \{\mathsf{0}, \mathsf{1}\}^* = (\textbf{alphabet}\, M)^* = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}}$, it follows that $w \in \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{D}}$. Thus there are two cases to consider:

- Suppose $w \in \Lambda_{\mathsf{A}}$. By part (A) of our induction on Λ, we have $w \in X_{\mathsf{ee}}$. Thus **zeros** w and **ones** w are both even. Since **ones** w is even, we have $w \notin X_{\mathsf{eo}}$. Since **zeros** w is even, we have $w \notin X_{\mathsf{oe}}$. Thus $w \notin X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$—contradiction.
- Suppose $w \in \Lambda_{\mathsf{D}}$. By part (D) of our induction on Λ, we have $w \in X_{\mathsf{oo}}$. Thus **zeros** w and **ones** w are both odd. Since **zeros** w is odd, we have $w \notin X_{\mathsf{eo}}$. Since **ones** w is odd, we have $w \notin X_{\mathsf{oe}}$. Thus $w \notin X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$—contradiction.

Because we obtained a contradiction in both cases, we have an overall contradiction. Thus $w \in L(M)$.

Question 4

Suppose, toward a contradiction, that X is regular. Thus there is an $n \in \mathbb{N} - \{0\}$ with the property of the Pumping Lemma for Regular Languages, where X has been substituted for L. Let $z = \mathsf{0}^{2n}\mathsf{1}^1\mathsf{2}^{2n}\mathsf{3}^1$. Because $2n + 1 = 2n + 1$, $2n$ is even, 1 is odd, $2n$ is even, and 1 is odd, we have that $z \in X$. And $|z| = 4n + 2 \geq n$. Thus the property of the pumping lemma tells us that there are $u, v, w \in \mathbf{Str}$ such that $z = uvw$ and

(1) $|uv| \leq n$; and

(2) $v \neq \%$; and

(3) $uv^iw \in X$, for all $i \in \mathbb{N}$.

Since $\mathsf{0}^{2n}\mathsf{1}^1\mathsf{2}^{2n}\mathsf{3}^1 = z = uvw$, (1) tells us that uv consists of only $\mathsf{0}$'s. Thus (2) tells us that $v = \mathsf{0}^p$ for some $p \geq 1$. Consequently, uw has: $2n - p$ occurrences of $\mathsf{0}$; 1 occurrence of $\mathsf{1}$; $2n$ occurrences of $\mathsf{2}$; and 1 occurrence of $\mathsf{3}$. Thus the sum of the numbers of occurrences of $\mathsf{0}$ and $\mathsf{1}$ in uw is $(2n-p)+1 = (2n+1)-p$, whereas the sum of the numbers of occurrences of $\mathsf{2}$ and $\mathsf{3}$ in uw is $2n + 1$. But according to (3), $uw = uv^0w \in X$, with the consequence that $(2n + 1) - p = 2n + 1$. But then $p = 0$—contradiction. Thus X is not regular.

Question 5

First, we convert the regular expression $(\mathsf{0}+\mathsf{1})^*$ generating $\{\mathsf{0},\mathsf{1}\}^*$ into a DFA, **allStrDFA**. (We first convert it to an FA, then to an EFA, then to an NFA, and then to a DFA.) Then $L(\textbf{allStrDFA}) = L((\mathsf{0}+\mathsf{1})^*) = \{\mathsf{0},\mathsf{1}\}^*$. Next, we concatenate α and β, calling the resulting regular expression λ. We then convert λ into a DFA, M. Thus $L(M) = L(\alpha)L(\beta)$. Let

$$Y = \{\, w \in \{\mathsf{0},\mathsf{1}\}^* \mid \text{there are } x, y \in \{\mathsf{0},\mathsf{1}\}^* \text{ such that } w = xy \text{ and } x \in L(\alpha) \text{ and } y \in L(\beta) \,\}.$$

It is easy to check that $L(M) = L(\alpha)\,L(\beta) = Y$. Next, let the DFA **ansDFA** = **minus**(**allStrDFA**, M). Then $L(\textbf{ansDFA}) = \{\mathsf{0},\mathsf{1}\}^* - Y = X$. Finally, we convert **ansDFA** into a regular expression, using our weak simplification function on regular expressions as the regular expression simplifier, producing γ. Hence $L(\gamma) = X$.